Geometry Round Problems — Sample Set

1. On a regular hexagon $ABCDEF$ with side length 2026^{2026}, a circle is drawn on each side such that the side is the diameter of its circle. Repeating this process for all six sides, the circles intersect at six points inside the hexagon. Connecting these points forms a new hexagon $A_1B_1C_1D_1E_1F_1$. Denote this transformation as X.
   The process X is applied repeatedly to the resulting hexagons, forming a sequence $A_iB_iC_iD_iE_iF_i$. Determine the largest number of times, a (counting the first transformation shown in the diagram), that X can be applied such that the side length of the resulting hexagon remains an integer. Compute a.

2. Circle Ω has diameter AB of length 10. There exists point C on the circumference of circle Ω such that segment $BC = 3$. The angle bisector of $\angle ACB$ is extended to hit circle Ω again at D. What is AD^2?

3. Square $ABCD$ has side length 10. The center of circle Ω is located within $ABCD$ such that circle Ω is tangent to both line segment AB and diagonal BD. The largest possible radius of the circle can be expressed as $a\sqrt{b} - c$ where a, b, c are integers and b is squarefree. What is $a + b + c$?

4. Triangle $\triangle ABC$ has side lengths $AB = 13$, $AC = 14$, $BC = 15$. Let M be the midpoint of side BC. Denote the cirumcircle of $\triangle ABC$ as Ω. There exists a circle ω which is internally tangent to Ω at point X and tangent to side BC at M, such that X and A lie on the same side of BC. Denote the center of ω as O. Lines AB and MX intersect at point Y. The value $\sin\angle BYO$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime. Compute $m + n$.

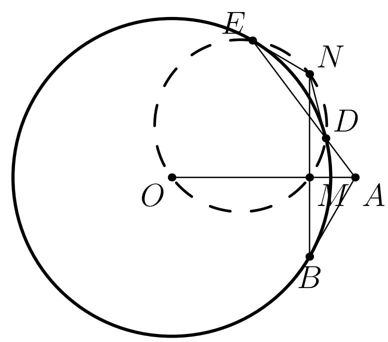
5. Let $ABCD$ be a convex quadrilateral with $AB \parallel CD$. Lines AD and BC are extended past A and B and intersect at point E. Given that $EA = 5$, $ED = 15$, $AB = 6$, and $EA \cdot ED = EB \cdot EC$, the circumradius of $\triangle ABC$ can be expressed as $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime and b is squarefree. Compute $a + b + c$.

6. A right square pyramid has a base with side length 24 and height $24\sqrt{2}$. A sequence of spheres $S_1, S_2, S_3, \ldots$ is constructed such that S_1 is tangent to the base of the pyramid and to all four lateral faces. For each $n \geq 2$, the sphere S_n is tangent to sphere S_{n-1} and to all four lateral faces, and the center of S_n is closer to the apex of the pyramid than the center of S_{n-1} is. The sum of the surface areas of all the spheres in the sequence can be expressed as $A\pi$. Find A.

7. In triangle $\triangle ABC$, the side lengths are $AB = 25$, $AC = 52$, and $BC = 63$, and the area of $\triangle ABC$ is 630. Let H be the orthocenter of $\triangle ABC$ and let M be the midpoint of side BC. Let ω be a circle that is tangent to BC at M and passes through H. ω interesects the altitude from A to BC at two points, H and a second point K. The length HK can be expressed as $\frac{m}{n}$, where m and n are relatively prime. Compute $m + n$.

8. In triangle $\triangle ABC$, let the side lengths be $AB = 13$, $BC = 14$, and $AC = 15$. Let I be the incenter of the triangle, and let the ray AI intersect the circumcircle of $\triangle ABC$ at a point M (where $M \neq A$).Let K be the midpoint of the segment IM. A line ℓ is drawn through K such that ℓ is perpendicular to the line AM. This line ℓ intersects the lines AB and AC at points P and Q, respectively. The area of $\triangle APQ$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime. Compute $m + n$.

9. Given a circle Γ with center O and radius 780, let A be a point such that $OA = 900$. The two tangents from A to Γ touch the circle at points B and C. A line through A intersects Γ at points D and E such that D lies between A and E, and $AD = 240$. Let M be the midpoint of BC. The circumcircle of $\triangle MDE$ intersects the line BC at M and another point N. Compute the length of MN.



10. Let $\triangle ABC$ be an acute triangle with side lengths $AB = 17$, $AC = 25$, and $BC = 28$. Let H be the orthocenter of $\triangle ABC$. Let E and F be the feet of the altitudes from B and C to the sides AC and AB, respectively. The line EF intersects the line BC at point P. The line AP intersects the circumcircle of $\triangle AEF$ at a second point X (where $X \neq A$). The value of AX^2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.